

Undergraduate Students' Conceptions of Mathematical Representations

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Four Ways To Represent a System

Linear System	Vector Equation	Augmented Matrix	Matrix Equation
$\begin{cases} 2x + y - z = 5 \\ 3x - y + 2z = -1 \\ x - y - z = 0 \end{cases}$	$x \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + z \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$	$\left[\begin{array}{ccc c} 2 & 1 & -1 & 5 \\ 3 & -1 & 2 & -1 \\ 1 & -1 & -1 & 0 \end{array} \right]$	$\begin{bmatrix} 2 & 1 & -1 \\ 3 & -1 & 2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$

Theories: Duval's (1999, 2006, 2017) and Mine

Duval's Three Sources of Incomprehension in Mathematics

1) Treatments, 2) Conversions, and 3) The need to recognize many representations as indicating the same mathematical entity.

The Cognitive Paradox

Duval's suggestion that although we only have (perceptual) access to mathematical objects through the symbol systems that we use to represent them, mathematics comprehension requires that we not confound the representations with the represented mathematical object.

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Duval's Constructs	My Constructs
<i>A mathematical object</i> is an abstract entity that we only have access to through a symbol system.	The set of quantitative relationships represented by a number of linear equations, the <i>quantitative system</i> , is the mathematical object.
<i>Transformations</i> are changes to a representation both within a register (symbol system) and between registers (symbol systems) including graphical ones.	<i>Translations</i> are changes to a representation both within and between analytic registers.
Treatments are "transformations" within a register.	Treatments are "translations" within a register.
Conversions are "transformations" between registers.	Conversions are "translations" between registers.
Congruence/Incongruence of a transformation takes into account two criteria: transparency and unit-by-unit translation.	I take transparency to mean visual similarity and units to be sub-pieces of algebraic notation.
Registers of representation are differing symbol systems in which representations can be expressed.	Each mathematical system is designated to be a register of representation. A vector space is an example of one register of representation; representations are formed from scalars and vectors.
"Transforming" in reverse is a relevant consideration.	"Translating" in reverse is a relevant consideration.
Distinguishing between the representation and the represented is fundamental to mathematics comprehension.	The quantitative system is clearly defined to be the represented mathematical object. All representations, including the linear systems representation, are given equal precedence.

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The Theory of Quantitative Systems designates a set of quantitative relationships as the object of study in the context of systems of equations. The four representations involved are given equal precedence, and acknowledgement is given to the differing nature of translating between two representations depending on the direction of translation. Equitizing the representations and taking reverse translations into account allows the identification of 12 possible translations between the representations.

Research Questions

- 1) What are undergraduate students' descriptions of the thing represented by four representations of a system?
- 2) How do undergraduate students account for translations between the four representations of a system?

Data Collection & Analysis

My methodological choices for this study included the use of one-on-one, task-based clinical interviews (Clement, 2000; Hunting, 1997) which were video and audio recorded and for which students' written work was collected. Ten participants were chosen on the basis of selection criteria applied to a pool of volunteers from two junior-level university applied linear algebra classes.

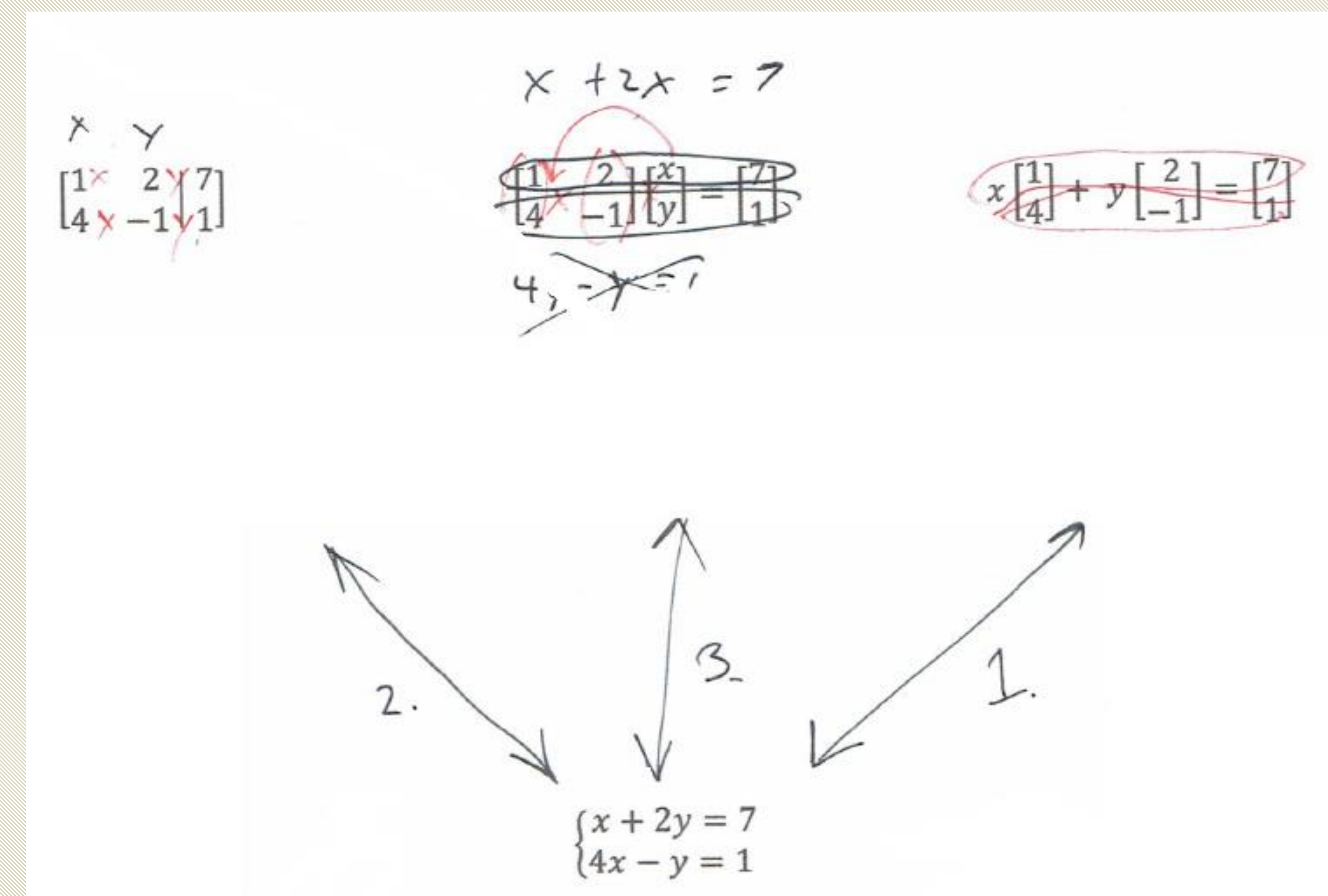
I conducted thematical analysis (Braun & Clarke, 2006) to explore the data (in transcript form) and categorize student responses. I also applied Duval's Theory of Semiotic Representation Registers (1997, 2006, 2017) in a new way and employed my newly posed Theory of Quantitative Systems.

Results for Question 1

"The Thing" Represented	An Exemplary Student Comment
System of Equations—Definitely	Felix (Student 3): All of this is basically just linear systems of equations.
System of Equations—Less Definitely	Jake (Student 10): Mentally I think of this (the linear system representation) ... as a baseline. But at the same time, I wouldn't say that this is definitively the baseline. ... I would say they're all reflections of each other.
Quantitative System	Myra (Student 7): (Referring to the linear system representation.) That's a representation of a mathematical truth or a mathematical statement that we learned first. That's the first way we learn to write it. These others are just different ways. ... Math is more than our representations of it. ... Even THIS isn't the thing! (She circles the graph of the system she created earlier in the interview.) It's a <i>graphical representation</i> of "the thing" (she does air quotes), whatever the thing is.
No Unified Thing	Ken (Student 9): I never really thought about it. I mentioned I think it's just notation. It doesn't really make a difference. I just kind of see a problem.

Results for Question 2

Students accounted for translations between the various representations by using visual techniques, heuristics, metaphors, and mathematical computation. Peter (Student 5) ranking of his difficulty with some of the translations as shown below where 1 indicates the easiest translation for him.



Implications

Analysis of the data led me to devise the constructs *enmeshed conception* and *enmeshed communication*. An *enmeshed conception* involves thinking where there is no distinction between representations and the entities they represent. *Enmeshed communication* involves speaking in ways that representations are indistinct from the mathematical entities they represent. The constructs provide potential explanations for the sometimes-startling degree of difficulty that students and instructors experience in linear algebra courses, particularly if experts' enmeshed communication promotes enmeshed conceptions amongst students.

I would say that I have identified a potential expert blind spot (Nathan, Koedinger, and Alibali, 2001) where experts' content knowledge prevents them from viewing the content in terms of students' development and learning processes. I continue to consider ways that mathematics and language get confounded, either in one's mind or in one's communication, and how that might account for the exodus of students (and/or level of student frustration) we observe in university level linear algebra courses.

References

- Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, 3(2), 77–101.
- Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A.E. Kelly & R. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 547–589). Erlbaum, NJ: Lawrence.
- Duval, R. (1999). Representation, vision and visualization: Cognitive functions in mathematical thinking. In F. Hitt and M. Santos (Eds.), *Proceedings of the 21st Annual Meeting of the North American Group of Psychology in Mathematics Education*, Vol. 1 (pp. 3-26). Cuernavaca, Morelos, Mexico: PME-NA.
- Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61(1), 103–131.
- Duval, R. (2017). *Understanding the mathematical way of thinking—The registers of semiotic representations*. Cham, Switzerland, Springer International Publishing.
- Hunting, R. P. (1997). Clinical interview methods in mathematics education research and practice. *The Journal of Mathematical Behavior*, 16(2), 145–165.
- Nathan, M. J., Koedinger, K. R., & Alibali, M. W. (2001). Expert blind spot: When content knowledge eclipses pedagogical content knowledge. In *Proceedings of the Third International Conference on Cognitive Science*.